Investigating Sicherman Dice

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1 Introduction

The concept of probability is a fascinating one, where one can find some amazing results stemming from some common items. Take for example the concept of two standard dice, cubes with faces numbered one through six. One would think that if would take the sum of the faces of the dice if rolled, you would have equally likely outcomes since each number has the same chance, or probability, of landing face up. However, if one observes the table of the sums, one would get the following results:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

This table is the standard sums for 2 dice, each numbered 1,2,3,4,5,6. We notice the sums s_i for $i = 1, 2, 3, ..., 11$ occurs a set number of times, where $s_1 = 2, s_2 = 3, ..., s_{11} = 12$.

	1	2	3	4	5	6
1	s_1	s_2	s_3	s_4	s_5	s_6
2	s_2	s_3	s_4	s_5	s_6	s_7
3	s_3	s_4	s_5	s_6	s_7	s_8
4	s_4	s_5	s_6	s_7	s_8	s_9
5	s_5	s_6	s_7	s_8	s_9	s_{10}
6	s_6	s_7	s_8	s_9	s_{10}	s_{11}

These probabilities and sums have been used in a variety of ways, from uses in board games to gambling games like craps. Although the concept of dice is not a new one, the idea of creating dice with different numbers on the faces of the dice was. A man named George Sicherman discovered a way to use different numbers on the faces of the dice to have the same probability than that of two standard dice Gardner [1978].It is because of his discovery we call the dice Sicherman dice. Below is a table of the labels of the dice and the probabilities:

	1	2	2	3	3	4
1	2	3	3	4	4	5
3	4	5	5	6	6	7
4	5	6	6	7	7	8
5	6	7	7	8	8	9
6	7	8	8	9	9	10
8	9	10	10	11	11	12

It is based on this discovery that questions were posed as to can this be expanded to find any other solutions, or is there a mathematical process to find them or prove there is not any other ones instead of simple trial and error. The most common way of proving Sicherman dice in this case involves generating functions and finding conditions that these functions are restricted to. This has led to the conclusion that there are only two solutions to labeling dice to obtain the standard probability of two dice, the trivial one being the standard dice, and that of Sicherman dice. It is based on this idea of relabeling of the faces of the dice that leads into this exploration of this paper. After reading articles detailing some of the initial methods of answering these questions, this article will answer whether or not the value on the dice makes any difference as to the number of configurations. The theorem we want to prove is the following:

Theorem 1 *Let two dice be labeled with positive integer values, one with a_i values, the other with b_i, where $i = 1, 2, 3, 4, 5, 6$. If we let for each $a_i \leq a_{i+1}$ and $b_i \leq b_{i+1}$, then there are only two ways to label the dice that has the same frequency table for sums of that of two standard dice. The solutions are the following:*

1. $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$ *and* $b_1 < b_2 < b_3 < b_4 < b_5 < b_6$

2. $a_1 < a_2 = a_3 < a_4 = a_5 < a_6$ *and* $b_1 < b_2 < b_3 < b_4 < b_5 < b_6$

2 Method for Proof

The structure of this proof comes from an article about finding forming arithmetic sums from the sums of the dice [Swift, 1999]. In the article, the idea is if we have sums with the same probabilities as standard dice sums, could one find a way to write out the numbers on the faces so that one could generate the probabilities given two integers to find the labels of the faces. Instead of assuming that each label on the face of the dice is strictly less than the next, the proof needs to include the possibility that a value could repeat. This leads into the structure of the proof and some observations.

Given two dice, one die labeled with a_i's where and the other die is labeled b_i's where $i = 1, 2, 3, 4, 5, 6$. Then we assume that $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6$ and $b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5 \leq b_6$. We are left to check if there exists a table where one could fill in the sums s_1 through s_{11} that follows the standard dice while meeting the conditions for the labels. This leads to checking the possibilities for what the inequality between labels can be, either strictly less than, or equal to the next value. This leads to some observations:

Observation 1: Since we need to have a smallest sum s_1, and a largest sum s_{11}, both occurring only one time, then this would require a_1 and b_1 to be the smallest values, and a_6 and b_6 be the largest values, which leads to $a_1 < a_2$ and $b_1 < b_2$ and $a_5 < a_6$ and $b_5 < b_6$.

Observation 2: In the creation of the table, one would notice that one could use the labels a_i's and b_i's interchangeably, meaning one could replace a's and b's.

Because of these two observations, we can limit the possibilities and fix the inequalities for a's and determine what the inequalities between b's would be. From observation 1, this means we only need to consider what inequality is between a_2, a_3, a_4, and a_5. This leads to the total possibilities for each in the table below.

	$a_1 < a_2$	$\square$	a_3	$\square$	a_4	$\square$	$a_5 < a_6$
I		$=$		$=$		$=$	
II		$=$		$=$		$<$	
III		$<$		$=$		$=$	
IV		$<$		$=$		$<$	
V		$<$		$<$		$=$	
VI		$=$		$<$		$<$	
VII		$=$		$<$		$=$	
VIII		$<$		$<$		$<$	

Now let's explore each case. Using the table, we will go through the cases in the order and inspect any case where the table can be filled in with the conditions that each sums can be placed. For each case, I will show the table filled in to a point where a contradiction occurs or if a completed table if possible.

2.1 Proof of Case I

	a_1	$<$	a_2	$=$	a_3	$=$	a_4	$=$	a_5	$<$	a_6
b_1	s_1										
$\wedge$											
b_2	s_2										
$\|$											
b_3	s_2										
b_4											
b_5											
$\wedge$											
b_6											s_{11}

The method for each of these cases is recognizing that we need a limited number of sums, $s_1, ..., s_{11}$ where each occurs so many times. In the table, we have a limitation where each sum greater than the previous one has to occur in a location that is either to the right or down from a smaller sum in the table. Understanding this, we know for every table that s_1 will always be the sum $a_1 + b_1$ and $s_{11} = a_6 + b_6$. The next sum to find would be s_2. Since $a_2 = a_3 = a_4 = a_5$, we can see that s_2 could not be placed to the right of s_1, thus this would require s_2 to be placed below s_1, resulting in $b_2 = b_3$ and $b_3 < b_4$. At this point, we can observe at some point, there is going to be a sum where
$s_k = a_2 + b_2 = a_2 + b_3 = a_3 + b_2 = a_3 + b_3 = a_4 + b_2 = a_4 + b_3 = a_5 + b_2 = a_5 + b_3$, which would mean we have a sum occurring eight times, which contradicts the condition that at most, a sum occurs six times. Therefore, no numbering of dice could have four values being the same.

2.2 Proof of Case II and Case III

	a_1	$<$	a_2	$=$	a_3	$=$	a_4	$=$	a_5	$<$	a_6
b_1	s_1		s_3		s_3		s_3		s_4		
$\wedge$											
b_2	s_2										
$\|$											
b_3	s_2										
$\wedge$											
b_4	s_4										
b_5	s_4										
$\wedge$											
b_6											s_{11}

Starting with s_2, we would clearly see that we cannot place s_2 to the right of s_1 as $a_2 = a_3 = a_4$, which would provide three s_2. The only possible way for this to work is if $b_2 = b_3$, and $b_3 < b_4$. This means that $s_2 = a_1 + b_2 = a_1 + b_3$. Next, we would have to place s_3 in the spots next to s_1 since $b_5 < b_6$ which results in not enough spaces for the s_3 to belong to. This results in $s_3 = a_2 + b_1 = a_3 + b_1 = a_4 + b_1$. Next would be to find s_4. The only spots that can be used for s_4 would be where $a_1 + b_4$, $a_1 + b_5$, $a_5 + b_1$ since $a_2 = a_3 = a_4$ and $b_2 = b_3$ resulting in the block of the table reserved for s_6. This means there is not enough places for s_4 and s_5 to occur, thus contradicting the conditions, meaning it is not possible for this configuration for the faces of the die to work. Now, an observation is made for the next case, where one does not have to do any additional to disprove the existence of the configuration. If we take the next case, Case III,

$a_1 < a_2 < a_3 = a_4 = a_5 < a_6$ and reverse the order, we would get $a_6 > a_5 = a_4 = a_3 > a_2 > a_1$. When we make the new table for this (and repeat for the b_i's, we would get the same conditions for placing sums in the table, but instead of greater sums needing to be placed either below or to the right of the previous sums, we get that sums for this rearrangement have to be less than to that the of the previous sums to the right or below. This leads to a type of symmetry to the cases for future considerations. We can follow the same steps used in Case II to disprove that Case III also is a contradiction by this observation.

2.3 Proof of Case IV

Since there are many possibilities, we will try to limit our choices by predetermining some inequalities for b_i's.

2.3.1 $b_2 = b_3$

	a_1	$<$	a_2	$<$	a_3	$=$	a_4	$<$	a_5	$<$	a_6
b_1	s_1		s_3		s_4		s_4		s_5		s_6
$\wedge$											
b_2	s_2		s_4		s_5		s_5		s_6		s_9
$\Vert$											
b_3	s_2		s_4		s_5		s_5		s_6		s_9
$\wedge$											
b_4	s_3		s_6		s_7		s_7		s_8		s_{10}
$\Vert$											
b_5	s_3		s_6		s_7		s_7		s_8		s_{10}
$\wedge$											
b_6	s_6		s_7		s_8		s_8		s_9		s_{11}

From the assumption that $b_2 = b_3$, we can fill in the table and actually make a possible solution where the correct number of sums occur for each consecutive sums. We will now try to find an explicit values using system of equations.

Let's start by finding values and solving for the system of equations. I am going to focus on finding values all in terms of the same value. To do so,I am going to look at the differences in sums. I am going to let d represent the difference in values of $s_2 - s_1$ and use that to solve the system. First

$$d = s_2 - s_1 = b_3 - b_1$$

This implies that $b_3 = b_1 + d$. If we look at the rows b_1 and b_2, then all of those differences are by the same value of d. This implies

$$s_6 - s_5 = s_5 - s_4 = s_4 - s_3 = d$$

This allows us to see that $a_3 = a_2 + d$. Also, if we combine these differences, we can get $s_6 - s_4 = 2d = b_4 - b_3$ which means that $b_4 = b_3 + d = b_1 + 3d$. Finally, if we look at $s_6 - s_3 = 3d = b_6 - b_3$ which finally lets $b_6 = b_3 + 3d = b_1 + 6d$. Now, if we look at ways of writing out s_7, we get the following:

$$s_7 = a_3 + b_4 = (a_2 + d) + (b_1 + 3d) = (a_2 + b_1) + 4d$$

$$s_7 = a_2 + b_6 = a_2 + (b_1 + 6d) = (a_2 + b_1) + 6d$$

Clearly we see a contradiction, thus this is not a viable solution to the table.

2.3.2 $b_2 < b_3 < b_4$

	a_1	$<$	a_2	$<$	a_3	$=$	a_4	$<$	a_5	$<$	a_6
b_1	s_1		s_2		s_3		s_3		s_4		s_5
$\wedge$											
b_2	s_2		s_4		s_5		s_5		s_6		
$\wedge$											
b_3	s_3		s_5		s_6		s_6				
$\wedge$											
b_4	s_4		s_6								
$\|$											
b_5	s_4										
$\wedge$											
b_6	s_5										s_{11}

So now we have a few subcases left, we want to narrow our choices. In this case, we are letting $b_2 < b_3 < b_4$ and try to fill in the table. It is straightforward to plug into s_1, s_2, and s_3. When we look to place s_4, there are only places in $a_2 + b_2$ and $a_5 + b_1$, and in order to fill in the other values for the sums, we would have to let $b_4 = b_5$. Once this happens, then s_5 has only certain places to be plugged in, but we encounter a problem for s_6. There is not enough places to place s_6, therefore this is not a viable option.

2.3.3 $b_2 < b_3 = b_4$

Finally, we look at the case that $b_2 < b_3 = b_4$. At this point, it is straightforward to plug in for the first few sums. When we get to s_5, we see that there is not enough places for s_5 to be places to appear five times. This leads to nonviable solution, thus eliminates case IV.

	a_1	$<$	a_2	$<$	a_3	$=$	a_4	$<$	a_5	$<$	a_6
b_1	s_1		s_2		s_3		s_3		s_5		
$\wedge$											
b_2	s_2		s_3		s_4		s_4				
$\wedge$											
b_3	s_4		s_5								
$\|$											
b_4	s_4		s_5								
$\wedge$											
b_5	s_5										
$\wedge$											
b_6											

2.4 Proof of Case V and VI

For this case, there are different sub cases to handle with this case, when $b_2 < b_3 < b_4$ and when $b_2 = b_3 < b_4$.

2.4.1 $b_2 < b_3 < b_4$

	a_1	$<$	a_2	$<$	a_3	$<$	a_4	$=$	a_5	$<$	a_6
b_1	s_1		s_2		s_3						
$\wedge$											
b_2	s_2		s_3								
$\wedge$											
b_3	s_3										
$\wedge$											
b_4											
b_5											
$\wedge$											
b_6											

At this point, we have three possibilities for s_4, and each will be addressed.

	a_1	$<$	a_2	$<$	a_3	$<$	a_4	$=$	a_5	$<$	a_6
b_1	s_1		s_2		s_3		s_4		s_4		s_6
$\wedge$											
b_2	s_2		s_3		s_4		s_5		s_5		
$\wedge$											
b_3	s_3		s_4		s_5		s_6		s_6		
$\wedge$											
b_4	s_5		s_6								
$\wedge$											
b_5	s_6										
$\wedge$											
b_6											

If we place s_4 in the spaces $a_2 + b_3 = a_3 + b_2 = a_4 + b_1 = a_5 + b_1$. This leads to s_5 to be placed in the only two available places in the table. This leads to only five places for s_6, which is not enough, thus causing a contradiction. The other option for s_5 is to have $b_4 = b_5$, which leads to another contradiction because there isn't enough spaces for s_7

	a_1	$<$	a_2	$<$	a_3	$<$	a_4	$=$	a_5	$<$	a_6
b_1	s_1		s_2		s_3		s_4		s_4		s_6
$\wedge$											
b_2	s_2		s_3		s_4		s_5		s_5		
$\wedge$											
b_3	s_3		s_4		s_5		s_6		s_6		
$\wedge$											
b_4	s_5		s_6								
$\parallel$											
b_5	s_5		s_6								
$\wedge$											
b_6	s_6										

If we place s_4 at $a_4+b_1, a_5+b_1, a_3+b_2, a_2+b_3$, we can start filling in values for s_5. This would lead to $b_4 = b_5$ so that enough s_5 could be placed. This results in s_6 being placed in the table accordingly. The problem is that finding enough s_7 is not possible, thus is placement of s_4 is not possible.

	a_1	$<$	a_2	$<$	a_3	$<$	a_4	$=$	a_5	$<$	a_6
b_1	s_1		s_2		s_3		s_4		s_4		
$\wedge$											
b_2	s_2		s_3								
$\wedge$											
b_3	s_3										
$\wedge$											
b_4	s_4										
$\parallel$											
b_5	s_4										
$\wedge$											
b_6											

This last sub case is where s_4 is placed in $a_4 + b_1, a_5 + b_1$ and then letting $b_4 = b_5$ allowing our other two s_4 locations at $a_1 + b_4$ and $a_1 + b_5$. At this point, this leads to another contradiction because there isn't enough places for s_5 to be placed.

2.4.2 $b_2 = b_3 < b_4$

	a_1	$<$	a_2	$<$	a_3	$<$	a_4	$=$	a_5	$<$	a_6
b_1	s_1		s_3		s_4		s_5		s_5		
$\wedge$											
b_2	s_2		s_4		s_5						
$\|$											
b_3	s_2		s_4								
$\wedge$											
b_4	s_3		s_5								
$\|$											
b_5	s_3		s_5								
$\wedge$											
b_6	s_4										

For the last sub case, we must assume that $b_2 = b_3 < b_4$. By this assumption, we have $s_2 = a_1 + b_2 = a_2 + b_3$, and it requires that $b_4 = b_5$ to have enough s_3 sums. It then forces s_4, but there are 3 pairs of sums for s_5 when only five is needed, so therefore this is a contradiction. This leads to Case V and Case VI not to be possible.

2.5 Proof of Case VII

At this point there are two cases to consider for the inequalities for b_i's, one that matches Case VII or when it is strictly less than for each b_i.

2.5.1 $b_2 = b_3 < b_4 = b_5$

	a_1	$<$	a_2	$=$	a_3	$<$	a_4	$=$	a_5	$<$	a_6
b_1	s_1		s_2		s_2						
$\wedge$											
b_2											
$\|$											
b_3											
$\wedge$											
b_4											
$\|$											
b_5											
$\wedge$											
b_6											

Immediately, we can see that there is a contradiction with this case. s_2 has to be placed in one of two ways, but since it's symmetrical, we can let $s_2 = a_2 + b_1 = a_3 + b_1$. When we search for s_3, there are two pairs for s_3, but since s_3 only needs three s_3, this is a contradiction.

2.5.2 $b_2 < b_3 < b_4 < b_5$

Finally, we are left with the Sicherman dice configuration

	a_1	$<$	a_2	$=$	a_3	$<$	a_4	$=$	a_5	$<$	a_6
b_1	s_1		s_2		s_2		s_3		s_3		s_4
$\wedge$											
b_2	s_3		s_4		s_4		s_5		s_5		s_6
$\wedge$											
b_3	s_4		s_5		s_5		s_6		s_6		s_7
$\wedge$											
b_4	s_5		s_6		s_6		s_7		s_7		s_8
$\wedge$											
b_5	s_6		s_7		s_7		s_8		s_8		s_9
$\wedge$											
b_6	s_8		s_9		s_9		s_{10}		s_{10}		s_{11}

2.6 Proof of Case VIII

And by going through all the cases, we are left with the standard configuration for two dice:

	a_1	$<$	a_2	$<$	a_3	$<$	a_4	$<$	a_5	$<$	a_6
b_1	s_1		s_2		s_3		s_4		s_5		s_6
$\wedge$											
b_2	s_2		s_3		s_4		s_5		s_6		s_7
$\wedge$											
b_3	s_3		s_4		s_5		s_6		s_7		s_8
$\wedge$											
b_4	s_4		s_5		s_6		s_7		s_8		s_9
$\wedge$											
b_5	s_5		s_6		s_7		s_8		s_9		s_{10}
$\wedge$											
b_6	s_6		s_7		s_8		s_9		s_{10}		s_{11}

3 Extension of Proof

Now that we have shown that there are only two configurations, the standard and Sicherman, we can find all possible sets of numbers that would fit one of these two configurations. First, let's look at the standard configuration.

3.1 Standard Dice Configuration

Notice that for each $a_i < a_{i+1}$ and $b_i < b_{i+1}$ for $i = 1, 2, 3, 4, 5$. Now, we can see that $s_1 = a_1 + b_1$ and $s_2 = a_2 + b_1 = a_1 + b_2$. If we look at $s_2 - s_1$:

$$d = s_2 - s_1 = (a_1 + b_2) - (a_1 + b_1) = b_2 - b_1$$

$$d = s_2 - s_1 = (a_2 + b_1) - (a_1 + b_1) = a_2 - b_1$$

By these equations, we can rewrite these equations in terms of a_1 and b_1 to obtain $a_2 = a_1 + d$ and $b_2 = b_1 + d$. The next step is to repeat this difference in sums and write out the values for a_i and b_i with respect to a_1 and b_1. So let's look at s_3. We know that $s_3 = a_1 + b_3 = a_2 + b_2 = a_3 + b_1$

$$s_3 = a_2 + b_2 = (a_1 + d) + (b_1 + d) = (a_1 + b_1) + 2d = s_1 + 2d$$

By finding that $s_3 = s_1 + 2d$, we can find what a_3 and b_3 will be. Since $s_3 = a_3 + b_1$, we can find that $a_3 = a_1 + 2d$ and similarly $b_3 = b_1 + 2d$.

The next value would be s_4. Since we know $s_4 = a_2 + b_3$, we can substitute what we have, and we get

$$s_4 = a_2 + b_3 = (a_1 + d) + (b_1 + 2d) = (a_1 + b_1) + 3d = s_1 + 3d$$

This allows us to find what the values of a_4 and b_4 will be. Since $s_4 = a_1 + b_1 + 3d = a_4 + b_1 = a_1 + b_4$, we can see that $a_4 = a_1 + 3d$ and $b_4 = a_1 + 3d$.

Next, we we will look at s_5 and find it in terms of s_1.

$$s_5 = a_3 + b_3 = (a_1 + 2d) + (b_3 + 2d) = (a_1 + b_1) + 4d = s_1 + 4d$$

This leads us to finding values for a_5 and b_5. Since $s_5 = a_1 + b_1 + 4d = a_5 + b_1 = a_1 + b_5$, we can see that $a_5 = a_1 + 4d$ and $b_5 = a_1 + 4d$.

Finally, if we look at s_6, we can find the values for a_6 and b_6.

$$s_6 = a_4 + b_3 = (a_1 + 3d) + (b_1 + 2d) = (a_1 + b_1) + 5d = s_1 + 5d$$

By using this equation, we can get that $a_1 + 5d = a_6 + b_1 = a_1 + b_6$ which leads to $a_6 = a_1 + 5d$ and $b_6 = a_1 + 5d$. This allows us to find any configuration of dice faces with standard probabilities. Instead of using $1, 2, 3, 4, 5, 6$ on our faces of the dice, we can use $a, a + d, a + 2d, a + 3d, a + 4d, a + 5d$ for integers a and d.

3.2 Sicherman Dice Configuration

Next we want to look at the Sicherman dice configuration and determine the values for a_i's and b_i's.

First we look at the values for $s_2 - s_1$ just like before.

$$d = s_2 - s_1 = (a_2 + b_1) - (a_1 + b_1) = a_2 - a_1$$

So now we can see that the difference for $s_2 - s_1$ is equal to the difference for $a_2 - a_1$. Now if we jump and look at the qualities $s_6 - s_5$.

$$s_6 - s_5 = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = b_5 - b_4 = b_4 - b_3 = b_3 - b_2$$

So we can see that $d = s_6 - s_5$ is the answer to all those differences. This provides us with a lot of the differences for a_i's and b_i's. So let's explore them all.

First, $a_2 = a_1 + d$. This then leads to $a_3 = a_2 + d = a_1 + 2d$, $a_4 = a_3 + d = a_1 + 3d$. In this regards, it let's us see that all $s_{i+1} - s_i$ will be a difference of d. We can use this to help solve for the values for b_i's. If we look at $s_3 - s_1$, we would see:

$$s_3 - s_1 = (s_3 - s_2) + (s_2 - s_1) = 2d = b_2 - b_1$$

This leads to $b_2 = b_1 + 2d$. Now using the information for $s_6 - s_5$, we would get the following results: $b_3 = b_2 + d = b_1 + 3d$, $b_4 = b_3 + d = b_1 + 4d$, $b_5 = b_4 + d = b_1 + 5d$. The last value to find would be b_6. If we look at $s_9 - s_7$, we get:

$$s_9 - s_7 = b_6 - b_5 = b_5 - b_3 = (b_1 + 5d) - (b_1 + 3d) = 2d$$

This leads to $b_6 = b_5 + 2d = b_1 + 7d$. So now we have the formula to find all configurations for Sicherman dice. As long as the first die has $a, a+d, a+d, a+2d, a+2d, a+3d$ and the second dice $b, b+2d, b+3d, b+4d, b+5d, andb+7d$ for integers b and d.

4 Generating Functions and Finding other Alternatives

For the next part, we will explore generating functions and how to use them to find dice with the same probability as the standard dice, but with different number of faces. Using the conditions from [Broline, 1979], we get some conditions to follow when finding generating functions for the standard dice. Let $f(x) = x^1 + x^2 + x^3 + x^4 + x^5 + x^6$, where the exponent represent the face of the die and the coefficient represents the number of occurrences on the die. We want to pick $g(x)$ and $h(x)$ such that the following conditions are met:

1. $f^2(x) = g(x)h(x)$
2. $f^2(0) = g(0)h(0) = 0$
3. $f^2(1) = g(1)h(1) = 36$

By using these conditions, we first want to explore what $f^2(x)$ can be factored into and find what the values of each factored term is when $f(0)$ and $f(1)$ are evaluated.

$$f^2(x) = (x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^2 = x^2(x+1)^2(x^2+x+1)^2(x^2-x+1)^2$$

Now we want to see what values can be obtained for $f(0)$ and $f(1)$

$$f^2(0) = (0)^2(1)^2(1)^2(1)^2$$

$$f^2(1) = (1)^2(2)^2(3)^2(2)^2$$

Now to find any configuration of dice, we need to pick $g(x)$ where $g(0) = 0$ and $g(1)|36$. For this, I found all possible factors of 36, which are 1,2,3,4,6,9,12,18,36 and found all possible configurations of the functions. Below I have made a list of all possible configurations of $g(x)$ and the corresponding $h(x)$ (by doing polynomial division of $f^2(x)$ by $g(x)$ once I found $g(x)$.

First, let's look at the trivial case when $g(1) = 1$ and $h(1) = 36$

$$g(x) = x$$
$$h(x) = x(x+1)^2(x^2+x+1)^2(x^2-x+1)^2$$

Next, we look at when $g(1) = 2$ and $h(1) = 18$

$$g(x) = x(x+1)$$
$$h(x) = x(x+1)(x^2+x+1)^2(x^2-x+1)^2$$

We continue with when $g(1) = 3$ and $h(x) = 12$, where we get multiple solutions. The first set of solutions are

$$g(x) = x(x^2+x+1)$$
$$h(x) = x(x+1)^2(x^2-x+1)^2(x^2+x+1)$$

The other possible solution would be

$$g(x) = x(x^2+x+1)(x^2-x+1)$$
$$h(x) = x(x+1)^2(x^2+x+1)(x^2-x+1)$$

And finally, we will look at when $g(1) = 4$ and $h(1) = 9$. This one has three possible solutions, the first being

$$g(x) = x(x+1)^2$$
$$h(x) = x(x^2+x+1)^2(x^2-x+1)^2$$

The next possible solution would be

$$g(x) = x(x+1)^2(x^2-x+1)$$
$$h(x) = x(x^2+x+1)^2(x^2-x+1)$$

The last possible solution would be

$$g(x) = x(x+1)^2(x^2-x+1)^2$$
$$h(x) = x(x^2-x+1)^2$$

Of course, we could look at the case when $g(x) = 6$ and $h(x) = 6$, but by now we know that we will find the standard dice configuration, as well as the configuration of Sicherman.

5 Conclusion and Further Work

From all the work provided, we can see that there are only two unique configurations for dice to provide the same probability as standard dice, the standard dice themselves or Sicherman dice, although one can now generate any pairs of dice with integer values that will have the same probability as standard dice. Looking at the idea of generating functions, further work could look at how we can try to combine the work of finding configurations of dice and alternate sized dice and see if there are any other possible solutions besides the ones found above.

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